Homework 2, Fall 2009: AWGN Channel, Hamming Codes and Syndrome Decoding

Homework assignment #2 is due on Thursday, September 24, 2009, in class.
Note: Show all your work in answering each question! Full credit will only be given for solutions that include all steps.

1. AWGN (Additive White Gaussian Noise) Channel:
   Answer the following questions about the AWGN channel shown in Figure 1. The noise $n$ is Gaussian-distributed, with zero mean and variance $\sigma_n^2$. This is written as $n \sim \mathcal{N}(0, \sigma_n^2)$. The transmitted signal $x$ is either -1 or 1; that is, $x \in \{-1, 1\}$.

   ![Figure 1: AWGN (Additive White Gaussian Noise) Channel with input $x$, noise $n$ and output $y$.](image)

   (a) Using Matlab, plot $p(n)$, the noise probability density function (PDF), for $\sigma_n^2 = 1$ and for $\sigma_n^2 = 4$. Plot both curves on a single graph so that you can compare them.

   (b) What is $p(y > 3|x = 1)$ for $\sigma_n^2 = 1$? For $\sigma_n^2 = 4$?
   
   Hint: Use the Matlab command `erfc`.

   (c) What is $p(y > 3)$ for $\sigma_n^2 = 1$? For $\sigma_n^2 = 4$?
   
   Hint: Use the Matlab command `erfc`.

   (d) Is $p(y > 3)$ larger than $p(y > 3|x = 1)$? Explain why or why not.

2. (7,4) Hamming code:
   We are using the Lin and Costello convention that the parity-check matrix $H$ is of the form $[I_m \ A^T]$, where $I_m$ is the $m \times m$ identity matrix. A systematic parity-check matrix $H$ for the (7,4) Hamming code is given below (on the next page).
\[ H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \]

(a) Find the systematic generator matrix \( G \) for the \((7,4)\) Hamming code from the systematic \( H \).

(b) If the data or information sequence \( u = [1 \ 0 \ 0 \ 1] \), what is the corresponding codeword \( v \)?

(c) If a codeword \( v = [1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0] \) is sent, but the noisy sequence \( r = [1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0] \) is received, what is the error sequence \( e \)?

(d) For the noisy received sequence \( r \) above, what is the corresponding syndrome \( s \) and the decoded codeword \( \hat{v} \)?

(e) A codeword \( v = [1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1] \) is sent, but the noisy received sequence \( r = [1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1] \) is received. What is the corresponding syndrome \( s \) and the decoded codeword \( \hat{v} \)? Is the decoded codeword correct? Explain why or why not.

3. Write down the distance spectrum, which is equivalent to the weight distribution, of the \((7,4)\) Hamming code.

4. Use the same \((7,4)\) Hamming code generator matrix \( G \) and parity-check matrix \( H \) as above. Answer the following questions, using the noisy received sequence \( r = [0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0] \). The original codeword \( v \) is not known.

   (a) The channel is a BSC with crossover probability \( \rho = .1 \). What is the probability that one bit of \( r \) is wrong? What is the probability that a particular bit (say bit 1) is wrong? What is the probability that two bits of \( r \) are wrong? What is the probability that a specific two-bit combination (say bit 1 and bit 7) is wrong?

   (b) What is the corresponding syndrome \( s \) and the decoded codeword \( \hat{v} \)? Is the decoded codeword the most likely codeword? Explain why or why not.

   (c) Suppose the actual codeword was \( v = [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1] \). Explain why this codeword was or was not chosen as the decoded codeword \( \hat{v} \).

5. **Graduate Problem:**

   (a) Why is the distance spectrum equivalent to the weight distribution for a linear block code?

   (b) Design an encoder for the \((7,4)\) Hamming code using logic gates and shift registers.

   (c) Both the \((7,4)\) Hamming code and the R3 (rate 1/3 repetition) code have minimum distance \( d_{\text{min}} = 3 \). Which code would you prefer to use in a commercial application, and why?